Sequence in a Metric Space

Definition:— Let (X,d) be a metric space. A sequence in the metric space is a function $x : \mathbb{N} \to X$. We denote a sequence as $\{x_n\}$, where $x_n = x(n)$.

Definition (convergence):— Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is said to be convergent if $\exists$ $x \in X$ such that for each $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t.

$$d(x_n, x) < \varepsilon, \quad \forall \; n \geq N.$$

In otherwords, for each $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$, s.t. $x_n \in B(x, \varepsilon)$, $\forall$ $n \geq N$.

Here x is said to be the limit of $\{x_n\}$ and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$ or simply $\lim x_n = x$ or $x_n \to x$.

Theorem:— In a metric space every convergent sequence has a unique limit.

Proof:— Let (X,d) be a metric space and $\{x_n\}$ be a convergent sequence in X. If possible, let $\{x_n\}$ convergence two points x and y.

Let $\varepsilon > 0$ be arbitrary. Then $\exists$ $N_1, N_2 \in \mathbb{N}$ s.t.

$$d(x_n, x) < \varepsilon/2, \quad \forall \; n \geq N_1 \quad \text{--- (1)}$$

$$d(x_n, y) < \varepsilon/2, \quad \forall \; n \geq N_2 \quad \text{--- (2)}$$

Let $N = \max\{N_1, N_2\}$.

$$\begin{aligned} \text{Now } d(x,y) &\leq d(x, x_n) + d(x_n, y) \\ &< \varepsilon/2 + \varepsilon/2, \quad \forall \; n \geq N \left[\begin{smallmatrix}\text{From (1)}\\ \text{and (2)}\end{smallmatrix}\right] \\ &= \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $d(x,y) = 0$

$\therefore$ $x = y$

Thus the limit is unique.

Definition (Bounded sequence):— In a metric space, a sequence is said to be bounded, if the range of the sequence is a bounded subset of the metric space.

Theorem:— In a Metric space, every convergent sequence is bounded.

Proof:— Let (X,d) be a metric space and $\{x_n\}$ be a convergent sequence in X s.t. $x_n \to x$ in X.

Thus for $\epsilon = 1$, $\exists\ N \in \mathbb{N}$ s.t.

$d(x_n, x) < 1, \quad \forall\ n \geq N.$

Let $K_1 = \max\{d(x_n, x) : 1 \leq n \leq N\}$

Let $M = \max\{1, K_1\}$

Thus $d(x_n, x) \leq M, \quad \forall\ n \in \mathbb{N}$

Now let $m, n \in \mathbb{N}$.

Now
$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) \leq M + M = 2M$$

Thus the diameter of the range of the sequence $\{x_n\}$ is bounded by $2M$.

Hence $\{x_n\}$ is bounded.

Theorem:— A sequence $\{x_n\}$ in a metric space (X,d) converges to a point $x \in X$ iff the sequence $\{d(x_n, x)\}$ of real numbers to zero. [i.e. $\lim x_n = x$ iff $\lim d(x_n, x) = 0$ in $\mathbb{R}$]

Proof:— ~~Let $\epsilon>0$ be arbitrary.~~

First let $\lim x_n = x$ in X.

Let $\epsilon > 0$ be arbitrary. Then $\exists\ N \in \mathbb{N}$ s.t.

$d(x_n, x) < \epsilon$, if $n \geq N$

$\Rightarrow |d(x_n, x) - 0| < \epsilon$, if $n \geq N$

Thus $\lim d(x_n, x) = 0$ in $\mathbb{R}$.

Thus $\{d(x_n, x)\}$ converges to 0 in $\mathbb{R}$.

Conversely, let $d(x_n, x) \to 0$ in $\mathbb{R}$.

Let $\varepsilon > 0$ be arbitrary. Then $\exists\ N \in \mathbb{N}$ s.t. $|d(x_n, x) - 0| < \varepsilon$, if $n \geq N$

$\Rightarrow d(x_n, x) < \varepsilon$, if $n \geq N$

[$\because$ Both $d(x_n, x), \varepsilon > 0$]

$\Rightarrow \lim x_n = x$

Theorem:— Let $\{x_n\}, \{y_n\}$ be sequences in a metric space (X, d) s.t. $x_n \to x$ in X. Then $y_n \to x$ in X iff $d(x_n, y_n) \to 0$ in $\mathbb{R}$.

Proof:— Let $y_n \to x$ in X.

Now $d(x_n, y_n) \leq d(x_n, x) + d(x, y_n)$, $\forall n \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary. Then $\exists\ N_1, N_2 \in \mathbb{N}$ s.t.

$d(x_n, x) < \varepsilon/2$, $\forall n \geq N_1$ — (1)

$d(y_n, x) < \varepsilon/2$, $\forall n \geq N_2$ — (2)

Let $N = \max\{N_1, N_2\}$

Thus for $n \geq N$ from (1) and (2), we have

$$d(x_n, y_n) \leq d(x_n, x) + d(y_n, y)$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\therefore d(x_n, y_n) \to 0$ in $\mathbb{R}$.

Conversely, let $d(x_n, y_n) \to 0$ in $\mathbb{R}$.

Let $\varepsilon > 0$ be arbitrary. Then $\exists\ N_1, N_2 \in \mathbb{N}$ s.t.

$d(x_n, x) < \varepsilon/2$, $\forall n \geq N_1$ — (1)

$d(x_n, y_n) < \varepsilon/2$, $\forall n \geq N_2$ — (2)

Let $N = \max\{N_1, N_2\}$

Then for $n \geq N$, from (1) and (2),

$$d(y_n, x) \leq d(y_n, x_n) + d(x_n, x)$$
$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon$$

$\therefore y_n \to x$ in X.

Definition (Cauchy Sequence):—

A sequence in a metric space (X,d) is said to be a Cauchy sequence in X if for any $\epsilon>0$, $\exists$ $N \in \mathbb{N}$ s.t.

$$d(x_m, x_n) < \epsilon, \quad \forall \; m,n \geq N.$$

i.e. $d(x_m, x_n) \to 0$ as $m, n \to \infty$.

Theorem:— Every convergent sequence is a Cauchy sequence

Proof:— Let $\{x_n\}$ be a convergent sequence in a metric space (X,d).

Let $x_n \to x \in X$.

Let $\epsilon > 0$ be arbitrary. Then $\exists$ $N \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon/2$, $\forall$ $n \geq N$.

Thus if $m, n \geq N$, then

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

Thus $\{x_n\}$ is a Cauchy sequence

Note:— In general every Cauchy sequence is not convergent in any metric space.

Theorem:— Every Cauchy sequence is bounded.

Proof:— Let $\{x_n\}$ be a Cauchy sequence in a metric space (X,d).

Now for $\epsilon = 1$, $\exists$ $N \in \mathbb{N}$ s.t.

$d(x_m, x_n) < 1$, $\forall$ $m, n \geq N$.

Let $K = \max\{d(x_i, x_j) : 1 \leq i, j < N\}$

Let $M = \max\{K, 1\}$

Then $\forall$ $m, n \in \mathbb{N}$, $d(x_m, x_n) < M$.

Thus $\{x_n\}$ is a bounded sequence.

Theorem:— A Cauchy sequence in a metric space (X,d) is convergent iff it has a convergent subsequence.

Proof:— If a Cauchy sequence is convergent, then obviously every subsequence of it converges to the same limit. i.e. it has a convergent subsequence.

Conversely, let $\{x_n\}$ be a Cauchy sequence in (X,d) and $\{x_{n_k} : n_1 < n_2 < \cdots\}$ be a convergent subsequence of it converging to a point $x \in X$.

Let $\epsilon > 0$ be arbitrary. Then $\exists$ $N_1 \in \mathbb{N}$ s.t.

$$d(x_{n_k}, x) < \epsilon/2, \quad \forall\, n_k \geq N_1 \qquad \text{—— (1)}$$

Again since $\{x_n\}$ is a Cauchy sequence, $\exists\, N_2 \in \mathbb{N}$ s.t. $d(x_m, x_n) < \epsilon/2, \ \forall\, m, n \geq N_2$ —— (2)

Let $N = \max\{N_1, N_2\}$

Then $\forall\, n \geq N$, from (1) and (2), we have

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \epsilon/2 + \epsilon/2 \quad [\because n_k \geq n \geq N] \\ &= \epsilon \end{aligned}$$

$\therefore$ $x_n \to x$ and hence $\{x_n\}$ is convergent.

Examples:—

1) Consider the metric space $C[0,1]$, the set of all ~~—~~ real valued continuous functions on $[0,1]$, with the metric d_1, defined by $d_1(f,g) = \int_0^1 |f(t)-g(t)|\,dt$, $\forall\, f,g \in C[0,1]$.

Consider the sequence $\{x_n\}$ in $C[0,1]$, where $x_n = e^{-nt}$, $\forall\, n \in \mathbb{N}$.

Now $d_1(x_n, 0) = \int_0^1 |e^{-nt} - 0|\, dt = \int_0^1 e^{-nt} dt$

$$= \frac{1}{n}(1 - e^{-nt})$$

$$\to 0 \text{ as } n \to \infty.$$

Thus $\{x_n\}$ is convergent with limit 0, the zero function.

2) consider the ~~same~~ metric space $C[0,1]$, ~~with~~ with the metric d_2, defined by, $d_2(f,g) = \max\limits_{0 \le t \le 1} \{|f(t) - g(t)|\}$

consider the sequence $\{x_n\}$, in $C[0,1]$, where $x_n = e^{-nt}$, $\forall\, n \in \mathbb{N}$.

Now $d_2(x_n, 0) = \max\limits_{0 \le t \le 1} |e^{-nt} - 0|$

$$= \max_{0 \le t \le 1} \left\{ \frac{1}{e^{nt}} \right\}$$

$$= 1$$

$$\to 1 \text{ as } n \to \infty.$$

Thus $\{x_n\}$ is convergent with limit function $f(t) = 1$, $\forall\, t \in [0,1]$.

Note:— Note that here we take same sequence $\{x_n\}$ ~~or~~ ~~same on~~, where $x_n = e^{-nt}$, $\forall\, n \in \mathbb{N}$ in same space ~~2~~ $C[0,1]$.

But in two cases the metrics are differents d_1 and d_2.

Also the limit functions in two cases are different.

Thus convergence and limits are depend ~~on~~ on the metrics.