COMPLEX VARIABLES: The Residue Theorem and Application

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1 Residues

Let $f(z)$ be single-valued and analytic function inside and on a simple closed curve C expect at the singularity $z = a$. According to Laurent series about $z = a$ given by [1]

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \tag{1}$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz \qquad\qquad n = 0, \pm1, \pm2, \cdots \tag{2}$$

In special case $n = -1$, we have

$$\oint_C f(z)\,dz = 2\pi i a_{-1} \tag{3}$$

We call a_{-1} the *residue* of $f(z)$ at $z = a$.

However, in case where $z = a$ is a pole of order k, then

$$a_{-1} = \lim_{z\to a}\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right] \tag{4}$$

1.1 Examples

Example 1: Find the residues of (a) $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$, (b) $f(z) = e^z \csc^2 z$, (c) $f(z) = \frac{2z+1}{z^2-z-2}$, (d) $f(z) = \left(\frac{z+1}{z-1}\right)^2$, (e) $\frac{\sin z}{z^2}$, (f) $\cot z$, (g) $\frac{z^2+4}{z^3+2z^2+2z}$ (h) $e^{-1/z}\sin(1/z)$ at all point its poles in the finite plane.

Answer: (a) $f(z)$ has a double pole at $z = -1$ and a simple poles at $z = \pm 2i$.
Residue at $z = -1$ is

$$\lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left[(z+1)^2\frac{z^2-2z}{(z+1)^2(z^2+4)}\right] = \lim_{z\to -1}\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2} = -\frac{14}{25}$$

Residue at $z = 2i$ is

$$\lim_{z\to 2i}\left[(z-2i)\frac{z^2-2z}{(z+1)^2(z^2+4)}\right] = \left[\lim_{z\to 2i}\frac{z^2-2z}{(z+1)^2}\right]\left[\lim_{z\to 2i}\frac{z-2i}{(z^2+4)}\right]$$

$$= \frac{-4-4i}{(2i+1)^2}\lim_{z\to 2i}\frac{1}{2z} = \frac{-4-4i}{(2i+1)^2}\frac{1}{2i} = \frac{7+i}{25}$$

Similarly, residue at $z = -2i$, we can get $\frac{7-i}{25}$.
(b) $f(z)$ has double pole at $z = m\pi$ where $m = 0, \pm1, \pm2, \cdots$. The residue at $z = m\pi$ is

$$\lim_{z\to m\pi}\frac{1}{1!}\frac{d}{dz}\left[(z-m\pi)^2\frac{e^z}{\sin^2 z}\right] = \lim_{z\to m\pi}\frac{e^z(z-m\pi)^2\sin^2 z + 2e^z(z-m\pi)\sin^2 z - 2e^z(z-m\pi)^2\sin z\cos z}{\sin^4 z}$$

Let $u = z - m\pi$ or $z = u + m\pi$

$$\therefore\quad \lim_{u\to 0} e^{u+m\pi}\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{\sin^3 u} = e^{m\pi}\left[\lim_{u\to 0} e^{u+m\pi}\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{\sin^3 u}\right]$$

$$= e^{m\pi}\Big[\lim_{u\to 0}\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{\sin^3 u}\frac{\sin^3 u}{u^3}\Big] \quad [\because \lim_{u\to 0}\frac{u}{\sin u} = 1]$$

$$= e^{m\pi}\Big[\lim_{u\to 0}\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{u^3}\Big]$$

Using L'Hospital's rule 3 times we get

$$e^{m\pi}\Big[\lim_{u\to 0}\frac{-2u^2\sin u - u^2\cos u - 6u\sin u + 10u\cos u + 6\sin u + 6\cos u}{6}\Big] = e^{m\pi}$$

(c) $f(z)$ has pole at $z = -1, 2$.
The residue at $z = -1$ is

$$\lim_{z\to -1}\Big[(z+1)\frac{2z+1}{z^2-z-2}\Big] = \lim_{z\to -1}\Big[\frac{2z+1}{z-2}\Big] = \frac{1}{3}$$

And the residue at $z = 2$

$$\lim_{z\to 2}\Big[(z-2)\frac{2z+1}{z^2-z-2}\Big] = \lim_{z\to 2}\Big[\frac{2z+1}{z+1}\Big] = \frac{5}{3}$$

(d) $f(z)$ has double pole at $z = 1$
The residue at $z = 1$ is

$$\lim_{z\to 1}\frac{1}{1!}\frac{d}{dz}\Big[(z-1)^2\big(\frac{z+1}{z-1}\big)^2\Big] = \lim_{z\to 1} 2(z+1) = 4$$

(e) $f(z)$ has double pole at $z = 0$.
The residue at $z = 0$ is

$$\lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\Big[z^2\frac{\sin z}{z^2}\Big] = 1$$

(f) $f(z)$ has pole at $z = m\pi$ where $m = 0, \pm 1, \pm 2, \cdots$. The residue at $z = m\pi$ is

$$\lim_{z\to m\pi}\Big[(z-m\pi)\frac{\cos z}{\sin z}\Big] = \lim_{z\to m\pi}\Big[\frac{\cos z - (z-m\pi)\sin z}{\cos z}\Big] = 1$$

(g) $f(z)$ has pole at $z = 0$ and $z = -1 \pm i$.
The residue at $z = 0$ is

$$\lim_{z\to 0}\Big[z\frac{z^2+4}{z^3+2z^2+2z}\Big] = \lim_{z\to 0}\frac{z^2+4}{z^2+2z+2} = 2$$

The residue at $z = -1+i$ is

$$\lim_{z\to -1+i}\Big[(z+1-i)\frac{z^2+4}{z^3+2z^2+2z}\Big] = \Big[\lim_{z\to -1+i}\frac{z^2+4}{z(z+1+i)}\Big] = \frac{2-i}{-1-i} = \frac{-1+3i}{2}$$

Similarly, the residue at $z = -1-i$ is $\frac{-1-3i}{2}$.
(h) Let $z = 1/u$

$$e^{-1/z}\sin(1/z) = e^{-u}\sin u = (1 - u + \frac{u^2}{2!} - \frac{u^3}{3!} + \cdots)(u - \frac{u^3}{3!} + \cdots) = u - u^2 + \cdots = \frac{1}{z} + \frac{1}{z^2} + \cdots$$

The residue is 1.

2 Residue Theorem

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C expect at the singularities $a, b, c, \cdots$ inside C, which have residue given by $a_{-1}, b_{-1}, c_{-1}, \cdots$. The residue theorem state that [1]

$$\boxed{\oint_C f(z)\,dz} = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) \tag{5}$$

2.1 Evaluation of Definite Integrals

The following types are most common in practice
1.

$$\int_{-\infty}^{\infty} F(x)\,dx, \quad \text{where } F(x) \text{ is a rotational function.}$$

Consider $\oint_C F(z)\,dz$ along a contour C consisting of the line along the x axis from $+R$ to $-R$ and the semicircle Γ above the x axis having this line as diameter. Then $R \to \infty$, If $F(z)$ is an even function, this can be used to evaluate $\int_0^\infty F(x)\,dx$.

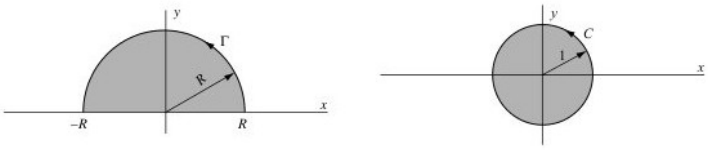

Figure 1: [1]

2.

$$\int_0^{2\pi} G(\sin\theta, \cos\theta)\,d\theta, \quad \text{where } G(\sin\theta,\cos\theta) \text{ is a rotational function of } \sin\theta \text{ and } \cos\theta.$$

Let $z = e^{i\theta}$. Then $\sin\theta = (z - z^{-1})/2i$, $\cos\theta = (z + z^{-1})/2$ and $dz = ie^{i\theta}$. The given integral is equivalent to $\oint_C F(z)\,dz$ where C is the unit circle with center at the origin.

3.

$$\int_{-\infty}^{\infty} F(x)\Big\{\cos mx \quad \text{or} \quad \sin mx\Big\}dx, \quad \text{where } F(x) \text{ is a rotational function.}$$

Here, we consider $\oint_C F(z)e^{imz}dz$ where C is the same contour as that in case 1.

2.2 Examples

Example 2: Evaluate (a) $\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$ around the circle with equation $|z| = 3$
(b) $\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)z^2}dz$ where C is (i) $|z-2i| = 6$,(ii) the square with vertices at $1+i, 2+i, 2+2i, 1+2i$.
(c) $\oint_C \frac{\sinh 3z}{(z-\pi i/4)^3}dz$ where C is square bounded by $x = \pm 2, y = \pm 2$.

Answer: (a) $f(z)$ has a double pole at $z = 0$ and two single pole at $z = -1 \pm i$. All poles are inside C.
The residue at $z = 0$ is

$$\lim_{z\to 0} \frac{1}{1!}\frac{d}{dz}\Big[z^2 \frac{e^{zt}}{z^2(z^2+2z+2)}\Big] = \lim_{z\to 0} \frac{te^{zt}(z^2+2z+2) - e^{zt}(2z+2)}{(z^2+2z+2)^2} = \frac{t-1}{2}$$

The residue at $z = -1+i$ is

$$\lim_{z\to -1+i} \Big[(z+1-i)\frac{e^{zt}}{z^2(z^2+2z+2)}\Big] = \lim_{z\to -1+i}\Big[\frac{e^{zt}}{z^2}\Big] \lim_{z\to -1+i}\Big[\frac{z+1-i}{z^2+2z+2}\Big]$$

$$= \frac{e^{(-1+i)t}}{-2i} \lim_{z\to -1+i}\Big[\frac{1}{2z+2}\Big] = \frac{e^{(-1+i)t}}{-2i}\frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Similarly, the residue at $z = -1-i$ is $\frac{e^{(-1-i)t}}{4}$.
By the residue theorem

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = 2\pi i\Big(\frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}\Big) = 2\pi i\Big(\frac{t-1}{2} + \frac{e^{-t}\cos t}{2}\Big)$$

(b) $f(z)$ has a triple pole at $z = -2$, double pole at $z = 0$ and two single pole at $z = \pm 2i$ (i) All poles are inside the circle C.
The residue at $z = -2$ is

$$\lim_{z\to -2} \frac{1}{2!}\frac{d^2}{dz^2}\Big[(z+2)^3 \frac{2z^2+5}{(z+2)^3(z^2+4)z^2}\Big] = \lim_{z\to -2} \frac{1}{2}\frac{d}{dz}\Big[-\frac{2(2z^2+5)}{z(z^2+4)^2} + \frac{4}{z(z^2+4)} - \frac{2(2z^2+5)}{z^3(z^2+4)}\Big]$$

$$= \lim_{z\to -2}\frac{1}{2}\Big[\frac{2(2z^2+5)}{z^2(z^2+4)^2}+\frac{8(2z^2+5)}{(z^2+4)^3}-\frac{8}{(z^2+4)^2}-\frac{4}{z^2(z^2+4)}-\frac{8}{(z^2+4)^2}+\frac{4(2z^2+5)}{z^2(z^2+4)^2}-\frac{8}{z^2(z^2+4)}+\frac{6(2z^2+5)}{z^4(z^2+4)}\Big]$$
$$= \frac{63}{256}$$

The residue at $z=0$ is

$$\lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\Big[z^2\frac{2z^2+5}{(z+2)^3(z^2+4)z^2}\Big] = \lim_{z\to 0}\Big[-\frac{2z(2z^2+5)}{(z+2)^3(z^2+4)^2}+\frac{4z}{(z+2)^3(z^2+4)}-\frac{3(2z^2+5)}{(z+2)^4(z^2+4)}\Big] = -\frac{15}{64}$$

The residue at $z=2i$ is

$$\lim_{z\to 2i}\Big[(z-2i)\frac{2z^2+5}{(z+2)^3(z^2+4)z^2}\Big] = \lim_{z\to 2i}\Big[\frac{2z^2+5}{(z+2)^3z^2}\Big]\lim_{z\to 2i}\Big[\frac{z-2i}{z^2+4}\Big] = -\frac{3}{128}(1+i)\frac{1}{4i} = \frac{3}{512}(i-1)$$

Similarly, the residue at $z=-2i$ is $\frac{3}{512}(-i-1)$
The required result is

$$2\pi i(\frac{63}{256}-\frac{15}{64}+\frac{3}{512}(i-1)+\frac{3}{512}(-i-1)) = 0$$

(ii) Only $z=2i$ pole is inside the square.
So, the residue at $z=2i$ is $\frac{3}{512}(i+1)$. Therefore the required result is $\frac{3\pi i}{256}(i-1)$

(c) $f(z)$ has a triple pole at $z=\pi i/4$ which in side the square.
The residue at $z=\pi i/4$ is

$$\lim_{z\to \pi i/4}\frac{1}{2!}\frac{d^2}{dz^2}\Big[(z-\pi i/4)^3\frac{\sinh 3z}{(z-\pi i/4)^3}\Big] = \lim_{z\to \pi i/4}\frac{9}{2}\sinh 3z = \frac{9i}{\sqrt{2}}$$

The required result is $2\pi i\times\frac{9i}{\sqrt{2}} = -\frac{9\pi\sqrt{2}}{2}$

Example 3: Evaluate (a) $\int_0^\infty \frac{dx}{x^6+1}$, (b) $\int_0^\infty \frac{dx}{x^4+1}$ (c) $\int_{-\infty}^\infty \frac{x^2}{(x^2+1)^2(x^2+2x+2)}dx$, (d) $\int_0^\infty \frac{dx}{(1+x^2)^2}$, (e) $\int_0^\infty \frac{x^2}{x^6+1}dx$, (f) $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2}$

Answer: **Definite integrals of the type** $\int_{-\infty}^\infty F(x)dx$

Let $|F(z)|\le M/R^k$ for $z=Re^{i\theta}$ where $k>1$ and M are constant. Let Γ is the semi-circle are of radius R shown in figure.

We know that

$$\Big|\int_\Gamma F(z)dz\Big| \le \frac{M}{R^k}\pi R = \frac{\pi M}{R^{k-1}}$$

Since the length of arc $L=\pi R$. Then

$$\lim_{R\to\infty}\Big|\int_\Gamma F(z)dz\Big| = 0 \quad \text{and so} \quad \lim_{R\to\infty}\int_\Gamma F(z)dz = 0$$

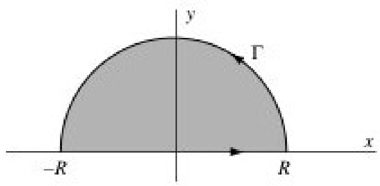

Figure 2: [1]

(a) Consider $\oint_C \frac{dz}{z^6+1}$, where C is the closed contour of Fig. 2 consisting of the line from $-R$ to R and the semi-circle Γ, traversed in the positive sense.
Since $z^6+1=0$, i.e., $z=e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, these are simple poles at $\frac{1}{z^6+1}$. Only the poles $e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}$ lie within C.
The residue at $z=e^{\pi i/6}$ is

$$\lim_{z\to e^{\pi i/6}}\Big[(z-e^{\pi i/6})\frac{1}{z^6+1}\Big] = \lim_{z\to e^{\pi i/6}}\frac{1}{6z^5} = \frac{e^{-5\pi i/6}}{6}$$

The residue at $z=e^{3\pi i/6}$ is

$$\lim_{z\to e^{3\pi i/6}}\Big[(z-e^{3\pi i/6})\frac{1}{z^6+1}\Big] = \lim_{z\to 3e^{\pi i/6}}\frac{1}{6z^5} = \frac{e^{-15\pi i/6}}{6}$$

The residue at $z = e^{\pi i/6}$ is

$$\lim_{z\to e^{5\pi i/6}} \Big[(z - e^{5\pi i/6})\frac{1}{z^6+1}\Big] = \lim_{z\to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{e^{-25\pi i/6}}{6}$$

Now

$$\oint_C \frac{dz}{z^6+1} = 2\pi i\Big[\frac{e^{-5\pi i/6}}{6} + \frac{e^{-15\pi i/6}}{6} + \frac{e^{-25\pi i/6}}{6}\Big] = \frac{2\pi}{3}$$

$$\Rightarrow \int_{-R}^{R} \frac{dx}{x^6+1} + \int_\Gamma \frac{dz}{z^6+1} = \frac{2\pi}{3}$$

The second term of the left hand side is zero as $R \to \infty$. We have

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} = \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

Since the integrand is even so

$$\int_0^\infty \frac{dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$$

(b) Consider $\oint_C \frac{dz}{z^4+1}$, where C is the closed contour of Fig. 2 consisting of the line from $-R$ to R and the semi-circle Γ, traversed in the positive sense.
Since $z^6 + 1 = 0$, i.e., $z = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}$, these are simple poles at $\frac{1}{z^4+1}$. Only the poles $e^{\pi i/4}, e^{3\pi i/4}$ lie within C.
The residue at $z = e^{\pi i/4}$ is

$$\lim_{z\to e^{\pi i/4}} \Big[(z - e^{\pi i/4})\frac{1}{z^4+1}\Big] = \lim_{z\to e^{\pi i/4}} \frac{1}{4z^3} = \frac{e^{-3\pi i/4}}{4}$$

The residue at $z = e^{3\pi i/4}$ is

$$\lim_{z\to e^{3\pi i/4}} \Big[(z - e^{3\pi i/4})\frac{1}{z^6+1}\Big] = \lim_{z\to e^{3\pi i/6}} \frac{1}{4z^3} = \frac{e^{-9\pi i/4}}{4}$$

Now

$$\oint_C \frac{dz}{z^4+1} = 2\pi i\Big[\frac{e^{-3\pi i/4}}{4} + \frac{e^{-9\pi i/4}}{4}\Big] = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_{-R}^{R} \frac{dx}{x^4+1} + \int_\Gamma \frac{dz}{z^4+1} = \frac{\pi}{\sqrt{2}}$$

The second term of the left hand side is zero as $R \to \infty$. We have

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^4+1} = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

Since the integrand is even so

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$

(c) Consider $\oint_C \frac{z^2}{(z^2+1)^2(z^2+2z+2)}\,dz$, where C is the closed contour of Fig. 2 consisting of the line from $-R$ to R and the semi-circle Γ, traversed in the positive sense.
The function has a pole at $z = \pm i$ and single pole at $-1 + \pm i$. Only the pole $z = i$ and $z = -1 + i$ lie within C.
The residue at $z = i$ is

$$\lim_{z\to i} \frac{d}{dz}\Big[(z-i)^2\frac{z^2}{(z^2+1)^2(z^2+2z+2)}\Big]$$

$$= \lim_{z\to i}\Big[-\frac{2z^2}{(z+i)^3\,(z^2+2z+2)} - \frac{(2z+2)z^2}{(z+i)^2\,(z^2+2z+2)^2} + \frac{2z}{(z+i)^2\,(z^2+2z+2)}\Big] = \frac{9i-12}{100}$$

Residue at $z = -1 + i$ is

$$\lim_{z\to -1+i} \Big[(z+1-i)\frac{z^2}{(z^2+1)^2(z^2+2z+2)}\Big] = \frac{3-4i}{25}$$

Then

$$\oint_C \frac{z^2}{(z^2+1)^2(z^2+2z+2)}\,dz = 2\pi i\Big[\frac{9i-12}{100}+\frac{3-4i}{25}\Big] = \frac{7\pi}{50}$$

or

$$\int_{-R}^{R} \frac{x^2}{(x^2+1)^2(x^2+2x+2)}dx + \int_\Gamma \frac{z^2}{(z^2+1)^2(z^2+2z+2)}\,dz = \frac{7\pi}{50}$$

Taking the limit as $R \to \infty$ and noting that the second integral approach zero and we obtain

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+2x+2)}dx = \frac{7\pi}{50}$$

(d) Consider $\oint_C \frac{dz}{(1+z^2)^2}$, where C is the closed contour of Fig. 2 consisting of the line from $-R$ to R and the semi-circle Γ, traversed in the positive sense. Here, $f(z) = \frac{1}{(1+z^2)^2}$

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{1}{z^3(1+\frac{1}{z^2})^2} = 0$$

$$\therefore \quad \lim_{R\to\infty} \int_\Gamma \frac{dz}{(1+z^2)^2} = 0$$

Since $f(z)$ has a double pole at $z = \pm i$. Only $z = i$ pole inside the contour C.
The residue at $z = i$ is

$$\lim_{z\to i} \frac{d}{dz}\Big[(z-i)^2\frac{1}{(z^2+1)^2}\Big] = \lim_{z\to i} \frac{-2}{(z+i)^3} = -\frac{i}{4}$$

$$\therefore \quad \oint_C \frac{dz}{(1+z^2)^2} = 2\pi i(-\frac{i}{4}) = \frac{\pi}{2}$$

$$\Rightarrow \quad \int_{-R}^{R} \frac{dx}{(1+x^2)^2} + \int_\Gamma \frac{dz}{(1+z^2)^2} = \frac{\pi}{2}$$

Taking the limit as $R \to \infty$ and noting that the second integral approach zero and we obtain

$$\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$$

(e) Consider $\oint_C \frac{z^2}{z^6+1}dz$, where C is the closed contour of Fig. 2 consisting of the line from $-R$ to R and the semi-circle Γ, traversed in the positive sense. Here, $f(z) = \frac{z^2}{z^6+1}$

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{1}{z^3(1+\frac{1}{z^6})} = 0$$

$$\therefore \quad \lim_{R\to\infty} \int_\Gamma \frac{z^2}{z^6+1}dz = 0$$

Since $f(z)$ has a pole at $z = e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, these are simple poles at $\frac{z^2}{z^6+1}$. Only the poles $e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}$ lie within C.
The residue at $z = e^{\pi i/6}$ is

$$\lim_{z\to e^{\pi i/6}} \Big[(z-e^{\pi i/6})\frac{z^2}{z^6+1}\Big] = \lim_{z\to e^{\pi i/6}}[z^2] \lim_{z\to e^{\pi i/6}} \Big[(z-e^{\pi i/6})\frac{1}{z^6+1}\Big] = \frac{1}{6}e^{-\pi i/2} = -\frac{i}{6}$$

The residue at $z = e^{\frac{3\pi i}{6}}$ is

$$\lim_{z\to e^{\frac{3\pi i}{6}}} \Big[(z-e^{3\pi i/6})\frac{z^2}{z^6+1}\Big] = \lim_{z\to e^{\frac{3\pi i}{6}}}[z^2] \lim_{z\to e^{\frac{3\pi i}{6}}} \Big[(z-e^{\frac{3\pi i}{6}})\frac{1}{z^6+1}\Big] = \frac{1}{6}e^{-\frac{9\pi i}{6}} = \frac{i}{6}$$

The residue at $z = e^{\frac{5\pi i}{6}}$ is

$$\lim_{z\to e^{\frac{5\pi i}{6}}} \Big[(z-e^{5\pi i/6})\frac{z^2}{z^6+1}\Big] = \lim_{z\to e^{\frac{5\pi i}{6}}}[z^2] \lim_{z\to e^{\frac{5\pi i}{6}}} \Big[(z-e^{\frac{5\pi i}{6}})\frac{1}{z^6+1}\Big] = \frac{1}{6}e^{-\frac{15\pi i}{6}} = -\frac{i}{6}$$

$$\therefore \quad \oint_C \frac{z^2}{z^6+1}dz = 2\pi i(-\frac{i}{6}+\frac{i}{6}-\frac{i}{6}) = \frac{\pi}{3}$$

$$\Rightarrow \quad \int_{-R}^{R} \frac{x^2}{x^6+1}dx + \int_{\Gamma} \frac{z^2}{z^6+1}dz = \frac{\pi}{3}$$

Taking the limit as $R \to \infty$ and noting that the second integral approach zero and we obtain

$$\int_{0}^{\infty} \frac{x^2}{x^6+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}dx = \frac{\pi}{6}$$

(f) Consider $\oint_C \frac{dz}{(z^2+1)(z^2+4)^2}$, where C is the closed contour of Fig. 2 consisting of the line from $-R$ to R and the semi-circle Γ, traversed in the positive sense.
The function has a pole at $z = \pm 2i$ and single pole at $\pm i$. Only the pole $z = 2i$ and $z = i$ lie within C. The residue at $z = 2i$ is

$$\lim_{z\to 2i} \frac{d}{dz}\Big[(z-2i)^2\frac{1}{(z^2+1)(z^2+4)^2}\Big] = \lim_{z\to 2i}\Big[-\frac{2z}{(z+2i)^2\,(z^2+1)^2} - \frac{2}{(z+2i)^3\,(z^2+1)}\Big] = \frac{11}{288}i$$

The residue at $z = ii$ is

$$\lim_{z\to 2i}\Big[(z-i)\frac{1}{(z^2+1)(z^2+4)^2}\Big] = -\frac{i}{18}$$

Then

$$\oint_C \frac{dz}{(z^2+1)(z^2+4)^2} = 2\pi i\Big[\frac{11}{288}i - \frac{i}{18}\Big] = \frac{5\pi}{144}$$

or

$$\int_{-R}^{R} \frac{dx}{(x^2+1)(x^2+4)^2} + \int_{\Gamma} \frac{dz}{(z^2+1)(z^2+4)^2} = \frac{5\pi}{144}$$

Taking the limit as $R \to \infty$ and noting that the second integral approach zero and we obtain

$$\int_{0}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{5\pi}{288}$$

Example 4: Evaluate (a) $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$, (b) $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ given $a > |b|$, (c) $\int_0^{2\pi} \frac{\sin 3\theta}{5-4\cos\theta}d\theta$, (d) $\int_0^{2\pi} \frac{\cos n\theta}{1+2a\cos\theta+a^2}d\theta$ and $\int_0^{2\pi} \frac{\sin n\theta}{1+2a\cos\theta+a^2}d\theta$, where $a^2 < 1$ and n is integer.

Answer:

Definite Integrals of the type $\int_0^{2\pi} G(\sin\theta, \cos\theta)d\theta$

Let, $z = e^{i\theta}$. Then $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i = (z - z^{-1})/2i$ and $\cos\theta = (e^{i\theta} + e^{-i\theta})/2 = (z + z^{-1})/2$, $dz = iz\,d\theta$

(a)

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_C \frac{dz/iz}{3-2(z+z^{-1})/2+(z-z^{-1})/2i}$$

$$= \oint_C \frac{2dz}{(1-2i)z^2+6iz-1-2i}$$

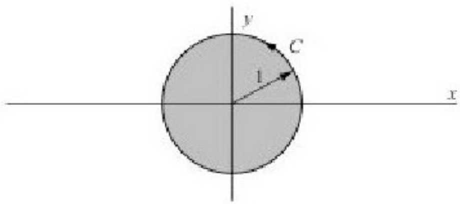

Figure 3: [1]

where C is the circle of unit radius with center at the origin, as shown in Fig.3.

The poles of $\frac{2}{(1-2i)z^2+6iz-1-2i}$ are the simple poles

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1-2i)(-1-2i)}}{2(1-2i)} = \frac{-6i \pm 4i}{2(1-2i)} = 2-i, (2-i)/5$$

Only $(2-i)/5$ lies inside C.

Residue at $z = (2-i)/5$ is

$$\lim_{z\to(2-i)/5}\Big[(z-(2-i)/5)\frac{2}{(1-2i)z^2+6iz-1-2i}\Big] = \lim_{z\to(2-i)/5}\Big[\frac{2}{2(1-2i)z+6i}\Big] = \frac{1}{2i}$$

Then

$$\oint_C \frac{2dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i\frac{1}{2i} = \pi$$

(b)

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_C \frac{dz/iz}{a+b(z+z^{-1})/2i} = \oint_C \frac{2dz}{bz^2+2aiz-b}$$

where C is the circle of unit radius with center at the origin, as shown in Fig.3.

The integrand has a pole at

$$z = \frac{-ai \pm \sqrt{a^2-b^2}i}{b}$$

Only $z_1 = \frac{-ai+\sqrt{a^2-b^2}i}{b}$ lies inside C.
Residue at z_1 is

$$\lim_{z\to z_1}\left[(z-z_1)\frac{2}{bz^2+2aiz-b}\right] = \lim_{z\to z_1}\frac{2}{2bz+2ai} = \frac{1}{bz_1+ai} = \frac{1}{\sqrt{a^2-b^2}i}$$

Then

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = 2\pi i\frac{1}{\sqrt{a^2-b^2}i} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

(c)

$$\int_0^{2\pi} \frac{\sin 3\theta}{5-3\cos\theta}d\theta = \oint_C \frac{\left(z^3-\frac{1}{z^3}\right)/\left(2i\right)}{5-3\left(z+\frac{1}{z}\right)/2}\frac{1}{iz}dz = \oint_C \frac{z^6-1}{2z^3\left(2z^2-5z+2\right)}dz$$

where C is the contour of Fig.3.

The integrand had a pole of order 3 at $z=0$ and a simple pole at $z=1/2$ inside C. The pole $z=2$ is outside the C.

Residue at $z=0$ is

$$\lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left[z^3\frac{z^6-1}{2z^3\left(2z^2-5z+2\right)}\right] = \frac{1}{4}\frac{d}{dz}\left[\frac{6z^5}{2z^2-5z+2} - \frac{(4z-5)\left(z^6-1\right)}{\left(2z^2-5z+2\right)^2}\right]$$

$$= \frac{1}{4}\left[-\frac{4\left(z^6-1\right)}{\left(2z^2-5z+2\right)^2} + \frac{2(4z-5)^2\left(z^6-1\right)}{\left(2z^2-5z+2\right)^3} - \frac{12(4z-5)z^5}{\left(2z^2-5z+2\right)^2} + \frac{30z^4}{2z^2-5z+2}\right] = -\frac{21}{16}$$

Residue at $z=1/2$ is

$$\lim_{z\to 1/2}\left[(z-1/2)\frac{z^6-1}{2z^3\left(2z^2-5z+2\right)}\right] = \frac{21}{16}$$

Then

$$\int_0^{2\pi}\frac{\sin 3\theta}{5-3\cos\theta}d\theta = 2\pi i(-\frac{21}{16}+\frac{21}{16}) = 0$$

(d)

$$\text{Let} \quad \int_0^{2\pi}\frac{e^{in\theta}}{1+a(e^{i\theta}+e^{-i\theta})+a^2}d\theta = \oint_C \frac{z^n}{1+a(z+\frac{1}{z})+a^2}\frac{dz}{iz} = \frac{1}{i}\oint_C \frac{z^n}{az^2+z\left(1+a^2\right)+a}dz$$

The integrand has pole at $z=-1,-\frac{1}{a}$. As $a^2<1$, so the pole at $z=-a$ is inside the contour ans $z=-\frac{1}{a}$ is outside the C. The residue at $z=-a$ is

$$\lim_{z\to -a}\left[(z+a)\frac{1}{i}\frac{z^n}{a(z+a)(z+\frac{1}{a})}\right] = \frac{1}{i}\frac{(-1)^n a^n}{(1-a^2)}$$

Then

$$I = 2\pi i\frac{1}{i}\frac{(-1)^n a^n}{(1-a^2)} = \frac{2\pi(-1)^n a^n}{(1-a^2)}$$

So the imaginary part is zero which yield

$$\int_0^{2\pi}\frac{\cos n\theta}{1+2a\cos\theta+a^2}d\theta = \frac{2\pi(-1)^n a^n}{(1-a^2)}, \quad \int_0^{2\pi}\frac{\sin n\theta}{1+2a\cos\theta+a^2}d\theta = 0$$

Evaluate 5: (a) $\int_0^\infty \frac{\cos mx}{x^2+1}dx, \quad m>0$, (b) $\int_{-\infty}^\infty \frac{x\sin\pi x}{x^2+2x+5}dx$

Answer: (a) Consider $\oint_C \frac{e^{imz}}{z^2+1}dz$ where C is the contour of Fig 2. The integrand has single pole at $z = \pm i$, but only $z = i$ lies inside C.
The residue at $z = i$ is

$$\lim_{z\to i}\left[(z-i)\frac{e^{imz}}{(z-i)(z+i)}\right] = \frac{e^{-m}}{2i}$$

Then

$$\oint_C \frac{e^{imz}}{z^2+1}dz = 2\pi i \frac{e^{-m}}{2i} = \pi e^{-m}.$$

$$\Rightarrow \int_{-R}^{R} \frac{\cos mx}{x^2+1}dx + i\int_{-R}^{R} \frac{\sin mx}{x^2+1}dx + \int_\Gamma \frac{e^{imz}}{z^2+1}dz = \pi e^{-m}$$

Taking the limit $R \to \infty$, the integrand around Γ approaches zero. and second part also zero. So

$$\int_0^\infty \frac{\cos mx}{x^2+1}dx = \frac{\pi}{2}e^{-m}$$

(b) Consider $\oint_C \frac{ze^{i\pi z}}{z^2+2z+5}dz$ where C is the contour of Fig 2. The integrand has single pole at $z = -1 \pm 2i$, but only $z = -1+2i$ lies inside C.
The residue at $z = -1+2i$ is

$$\lim_{z\to -1+2i}\left[(z+1-2i)\frac{ze^{i\pi z}}{z^2+2z+5}\right] = \frac{(-1+2i)e^{-\pi(i+2)}}{4i}$$

Then

$$\oint_C \frac{ze^{i\pi z}}{z^2+2z+5}dz = 2\pi i \frac{(-1+2i)e^{-\pi(i+2)}}{4i} = -\frac{\pi}{2}(-1+2i)e^{-2\pi}$$

$$\Rightarrow \int_{-R}^{R} \frac{xe^{imx}}{x^2+2x+5}dx + \int_\Gamma z\frac{e^{i\pi z}}{z^2+2z+5}dz = -\frac{\pi}{2}(-1+2i)e^{-2\pi}$$

Taking the limit $R \to \infty$, the integrand around Γ approaches zero. Equa the real and imaginary part we get

$$\int_{-\infty}^{\infty} \frac{x\sin\pi x}{x^2+2x+5}dx = -\pi e^{-2\pi}$$

Example 6: Show that $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$
Answer: (a) The function $f(z) = \frac{e^{iz}}{z}$ has a single pole at $z = 0$ and we cannot integrate through a singularity, so we modify the contour as shown in Fig. 4.

Since $z = 0$ is outside C.
We have

$$\oint_C \frac{e^{iz}}{z}dz = 0$$

$$\Rightarrow \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}dx + \int_{HJA} \frac{e^{iz}}{z}dz + \int_\epsilon^R \frac{e^{ix}}{x}dx + \int_{BDEFG} \frac{e^{iz}}{z}dz = 0$$

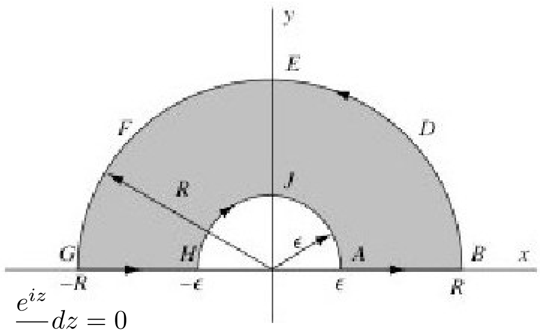

Figure 4: [1]

Replacing x by $-x$ in the first integral and combining with third integral, we find

$$\int_\epsilon^R \frac{e^{ix}-e^{-ix}}{x}dx + \int_{HJA} \frac{e^{iz}}{z}dz + \int_{BDEFG} \frac{e^{iz}}{z}dz = 0$$

$$\Rightarrow \quad 2i\int_\epsilon^R \frac{sinx}{x}dx = -\int_{HJA} \frac{e^{iz}}{z}dz - \int_{BDEFG} \frac{e^{iz}}{z}dz = 0$$

Let $\epsilon \to 0$ and $R \to \infty$, the second integral on the right side approaches zero. Letting $z = \varepsilon e^{i\theta}$ in the first integral on the right side is

$$\lim_{\epsilon \to 0} \int_{\pi}^{0} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta = \pi i$$

Then we have

$$\lim_{R\to\infty, \epsilon\to 0} 2i \int_{\epsilon}^{R} \frac{\sin x}{x} dx = \pi i \quad \text{or} \quad \boxed{\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}}$$

References

[1] M. R. Spiegel, S. Lipschutz, J. J. Schiller and D. Spellman, Complex Variables, Schaum's Outlines, ISBN-13: 978-0-07-008385-1