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Semester - IV
Subject - Linear Algebra
Paper - BMH4CC10
Topic - Linear Transformation.

Linear Transformation: 05/04/2020

Let V and W be vector space over the same field, say, F. A linear transformation from V to W is a function T from V to W s.t.

$$T(cx+y) = cT(x) + T(y), \quad \forall\, x, y \in V$$

and $c \in F$.

Examples

1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x_1, x_2) = (x_2, x_1), \quad \text{let } (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$$

and $c \in F$.

Now,
$$T(c(x_1, x_2) + (y_1, y_2)) = T(cx_1 + y_1, cx_2 + y_2)$$
$$= (cx_2 + y_2, cx_1 + y_1)$$
$$= c(x_2, x_1) + (y_2, y_1)$$
$$= cT(x_1, x_2) + T(y_1, y_2)$$

i.e. T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

2. The identity transformation defined on a vector space V, is a linear transformation from V to V. Also the zero transformation is a linear transformation from V to V.
i.e. $I\alpha = \alpha$ & $0\alpha = 0$, $\forall \alpha \in V$ are L.T.

3. Let F be a field and let V be the space of polynomial function f from F to F, given by

$f(x) = c_0 + c_1 x + \cdots + c_k x^k$.

let $(Df)(x) = c_1 + 2c_2 x + \cdots + k c_k x^{k-1}$.

let $f, g \in V$ & $c \in F$.

Now $cf(x) + g(x)$

$= c(c_0 + c_1 x + \cdots + c_k x^k) + (d_0 + d_1 x + \cdots + d_n x^n)$

Case I. let $k < n$, then

$cf(x) + g(x) = (cc_0 + d_0) + (cc_1 + d_1)x + \cdots + (cc_k + d_k)x^k + \cdots + d_n x^n$.

$\therefore D(cf(x) + g(x))(x) = (cc_1 + d_1) + 2(cc_2 + d_2)x + \cdots + k(cc_k + d_k)x^{k-1} + \cdots + n d_n x^{n-1}$

$= c(c_1 + 2c_2 x + \cdots + k c_k x^{k-1}) + (d_1 + 2d_2 x + \cdots + k d_k x^{k-1} + \cdots + n d_n x^{n-1})$

$= c(Df)(x) + (Dg)(x)$

Case II

Let $k \geq n$, then,

$$cf(x)+g(x) = (cc_0+d_0) + (cc_1+d_1)x + \cdots + (cc_n+d_n)x^n + \cdots + cc_kx^k$$

$\therefore$ ~~$D(cf(x)+g(x))(x)$~~

$\therefore D(cf+g)(x)$

$$= c(Df)(x) + (Dg)(x).$$

$\therefore D$ is a linear transformation from V to V.

Note 1. If T is a l.t. from V to W, then $T(0)=0$

since $T(0) = T(0+0) = T(0)+T(0)$.

$\Rightarrow T(0)=0$.

Note 2. A linear transformation from $\mathbb{R} \to \mathbb{R}$ is a straight line passing through the origin (according to our definition).

Note 3 A linear transformation 'preserves' linear combination i.e. $T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n)$

$$= c_1T(\alpha_1) + c_2T(\alpha_2) + \cdots + c_nT(\alpha_n).$$

$\alpha_1, \ldots, \alpha_n \in V$, $c_1, \ldots, c_n \in F$.

Thm Let V be a vector space over F. Let $\dim V = n$ and $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a ordered basis for V. Let W be another vector space over the same field F and $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be any vectors in W. Then $\exists$ unique linear transformation $T : V \to W$ s.t.

$$T\alpha_j = \beta_j \quad , \quad j = 1, 2, \ldots, n.$$

Proof Let $\alpha \in V$, then $\exists$ $c_1, c_2, \ldots, c_n \in F$ s.t.

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n.$$

Now let us define

$$T\alpha = c_1\beta_1 + \ldots + c_n\beta_n \longrightarrow (1)$$

Then T is well defined. Since T associating with each $\alpha \in V$ a vector $T\alpha \in W$.

$\therefore$ from (1) it is clear that $T\alpha_j = \beta_j$ for $j = 1, 2, \ldots n$

Let us now try to prove that T is linear.

Let $\beta \in V$, then $\exists$ $d_1, d_2, \ldots, d_n \in F$ s.t.

$$\beta = d_1\alpha_1 + \ldots + d_n\alpha_n.$$

Let $c \in F$.

$$\therefore c\alpha + \beta = (cc_1 + d_1)\alpha_1 + \ldots + (cc_n + d_n)\alpha_n.$$

$$\therefore T(c\alpha + \beta) = (cc_1 + d_1)\beta_1 + \ldots + (cc_n + d_n)\beta_n.$$

[by (1)]

Again, $c(T\alpha) + T\beta$

$$= c(c_1\beta_1 + \cdots + c_n\beta_n) + (d_1\beta_1 + \cdots + d_n\beta_n)$$

$$= (cc_1 + d_1)\beta_1 + \cdots + (cc_n + d_n)\beta_n.$$

$\therefore T(c\alpha + \beta) = c(T\alpha) + T\beta$

i.e. T is linear.

Uniqueness

Let T_1 be another L.T from $V \longrightarrow W$ s.t.

$$T_1(\alpha_j) = \beta_j, \text{ for } j = 1, 2, \cdots, n.$$

Then for vector $\alpha = \sum_{i=1}^{n} c_i \alpha_i$, we have

$$T_1\alpha = T_1\left(\sum_{i=1}^{n} c_i \alpha_i\right)$$

$$= c_1 T_1\alpha_1 + \cdots + c_n T_1\alpha_n$$

$$= c_1\beta_1 + \cdots + c_n\beta_n.$$

$\Rightarrow$ T_1 is exactly the same rule T.

i.e. the linear transformation is unique. Hence the result.

Def^n Let $T: V \to W$ be a linear transformation. V & W are the v.s over the same field F. The null space of T is the set defined as $\{\alpha \in V \,/\, T\alpha = 0\}$.

Rank of T

Let $T: V \to W$ a linear transformation. Let $\dim V = n$. Then rank of T is denoted by rank$T$ and defined as

$$\text{rank}\, T = \dim(\text{range of } T).$$

Nullity of T.

Let $T: V \to W$ a linear transformation. Let $\dim V = n$.

Now nullity of $T = \dim(\text{null space of } T)$.

Note: The null space of T is a subspace of V.

Since $T(0) = 0$ i.e. $0 \in$ null space of T.

$T(c\alpha_1 + \alpha_2) = cT(\alpha_1) + T(\alpha_2)$; $\alpha_1, \alpha_2 \in$ null space of T

$= c \cdot 0 + 0 = 0$

i.e. $c\alpha_1 + \alpha_2 \in$ null space of T. Hence the result.

13/04/2020

13/04/2020

Theorem

Let V and W be ~~the~~ vector space over F. Also let $\dim V = n$. $T: V \to W$ a linear transformation then,

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

Proof

Let us denote

$$\text{nullity}(T) = N(T)$$
$$\text{Rank}(T) = R(T).$$

Let $\{\alpha_1, \ldots \alpha_k\}$ a basis for the null space of T and since $\dim V = n$ $\exists$ vector $\alpha_{k+1}, \ldots, \alpha_n$ s.t. $\{\alpha_1, \ldots, \alpha_n\}$ a basis for V.

Now clearly $\{T\alpha_1, \ldots, T\alpha_n\}$ span range of T. Now since $T\alpha_i = 0$, $i = 1, \ldots k$, actually $T\alpha_{k+1}, \ldots, T\alpha_n$ spans range of T.

Now we show that $T\alpha_{k+1}, \ldots, T\alpha_n$ are linearly independent.

For scalar $c_{k+1}, \ldots, c_n$ we have

$$\sum_{j=k+1}^{n} c_j T\alpha_j = 0.$$

$\Rightarrow T\left(\sum_{j=k+1}^{n} c_j \alpha_j\right) = 0$ [$\because$ T is linear]

$\Rightarrow \sum_{j=k+1}^{n} c_j \alpha_j \in$ null space of T

Let $\alpha = \sum_{j=k+1}^{n} c_j \alpha_j$.

Again since $\{\alpha_1, \ldots \alpha_k\}$ - a basis for the null space of T, we have

$\alpha = \sum_{i=1}^{k} d_i \alpha_i$

$\therefore \sum_{i=1}^{k} d_i \alpha_i - \sum_{j=k+1}^{n} c_j \alpha_j = 0$.

Now since $\{\alpha_1, \ldots \alpha_n\} \rightarrow$ basis for V

$d_1 = \ldots = d_k = 0 = c_{k+1} = \ldots = c_n$.

$\Rightarrow T\alpha_{k+1}, \ldots, T\alpha_n$ are l.i.

i.e $R(T) = n - k$.

$N(T) = k$.

$\Rightarrow R(T) + N(T) = \dim V$

Hence the theorem.

Thm If A is an $m \times n$ matrix over the field F, then
row rank(A) = clounm rank(A).

Prove it.

Problem: Is there a L.T. T from $\mathbb{R}^3$ to $\mathbb{R}^2$ s.t. $T(1,-1,1) = (1,0)$ and $T(1,1,1) = (0,1)$?

Soln NO.
Since $\dim \mathbb{R}^3 = 3$ and
$\{(1,-1,1), (1,1,1)\}$ is not basis of $\mathbb{R}^3$.
Moreover, if
$(x,y,z) \in \mathbb{R}^3$ and
$(x,y,z) = c_1(1,-1,1) + c_2(1,1,1) + c_3(0,0,1)$
Let T be a L.T from $\mathbb{R}^3$ to $\mathbb{R}^2$
Now
$$T(x,y,z) = c_1 T(1,-1,1) + c_2 T(1,1,1) + c_3 T(0,0,1)$$
$$= c_1(1,0) + c_2(0,1) + c_3 T(0,0,1).$$
Then we need $T(0,0,1) = ?$
and here the result.

Problem: If $\alpha_1 = (1,-1)$, $\alpha_2 = (2,-1)$, $\alpha_3 = (-3,2)$ and $\beta_1 = (1,0)$, $\beta_2 = (0,1)$, $\beta_3 = (1,1)$ is there a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T\alpha_i = \beta_i$ for $i = 1, 2$ and 3?