Fourier Transforms

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1 Fourier Transforms

The Fourier transform provides a representation of functions defined over an infinite interval and having no particular periodicity, in terms of a superposition of sinusoidal functions [1]. It may thus be considered as a generalization of the Fourier series representation of periodic functions.

Now One can developed the transition from Fourier series to Fourier transforms. A function of period T may be represented as a complex Fourier series as

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi int/T} = \sum_{n=-\infty}^{\infty} a_n e^{i\omega_n t} \qquad \because \quad \omega_n = 2\pi n/T \tag{1}$$

As the period T tends to infinity, the frequency $\Delta\omega = \frac{2\pi}{T}$ becomes vanishingly small and the spectrum of allowed frequencies ω_n becomes a continuum. Thus, the infinite sum of terms in the Fourier series becomes an integral, and the coefficients a_n become functions of the continuous variable ω. The coefficients a_n is

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-2\pi int/T}dt = \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)e^{-i\omega_n t}dt \tag{2}$$

$$\therefore \quad f(t) = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\left(\int_{-T/2}^{T/2} f(u)e^{-i\omega_n u}du\right)e^{i\omega_n t} \tag{3}$$

If T tends to infinite then $\Delta\omega = \frac{2\pi}{T}$ becomes infinitesimal and we define

$$\sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \mathrm{g}(\omega_n)e^{i\omega_n t} \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{g}(\omega)e^{i\omega t}d\omega \tag{4}$$

In this case

$$\mathrm{g}(\omega_n) = \int_{-T/2}^{T/2} f(u)e^{-i\omega_n u}du \tag{5}$$

and $f(t)$ becomes

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}d\omega \int_{-\infty}^{\infty} f(u)e^{-i\omega u}du \tag{6}$$

This result is known as Fourier's inversion theorem.

Now we define the Fourier transform of $f(t)$ by

$$\mathrm{g}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt, \tag{7}$$

and its inverse by

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{g}(\omega)e^{i\omega t}d\omega. \tag{8}$$

The Fourier transformation is valid when the following conditions are satisfied [2]

1. The function $f(x)$ must be single valued function of the real variable x throughout the range $-\infty < x < \infty$. It may however have a finite number of finite discontinuous.

2. The integral $\int_{-\infty}^{\infty} |f(x)| dx$ must exist.

From the equation (7), we obtain

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos(\omega t)\, dt \tag{9}$$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin(\omega t)\, dt \tag{10}$$

These formulas define the Fourier cosine and Fourier sine transformation. It can be used to calculate the Fourier transformation for even and odd function.

1.1 Examples

1. Find the Fourier transform of $f(t) = e^{-\alpha|t|}$, with $\alpha > 0$. Hence evaluate the integral $\int_0^\infty \frac{\cos(\omega t)}{\alpha^2+\omega^2}\, d\omega$

$$\mathrm{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{\alpha t - i\omega t} dt + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\alpha t - i\omega t} dt$$
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} \right] = \frac{1}{\sqrt{2\pi}} \frac{2\alpha}{\alpha^2+\omega^2}$$

Using the inverse Fourier transfer, we get

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{2\alpha}{\alpha^2+\omega^2} e^{i\omega t} d\omega$$

For the interval $(0, \infty)$, t is positive, so $f(t) = e^{-\alpha|t|} = e^{-\alpha t}$. If we take the real part and we get

$$\int_0^\infty \frac{\cos(\omega t)}{\alpha^2+\omega^2}\, d\omega = \frac{\pi}{2\alpha} e^{-\alpha t}$$

2. Find the Fourier transform of $f(t) = \delta(t)$.

$$\mathrm{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}}$$

3. Find the Fourier transform of $f(x) = Ae^{-t^2/2\sigma^2}$.

$$\mathrm{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Ae^{-t^2/2\sigma^2} e^{-i\omega t} dt = A\sigma e^{-\omega^2\sigma^2/2}.$$

$$\left[\text{Hints: } \int_{-\infty}^{\infty} e^{-\alpha x^2 + \beta x} dx = e^{\beta^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(x-\beta/2\alpha)^2} dx = e^{\beta^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha z^2} dz = e^{\beta^2/4\alpha} \sqrt{\frac{\pi}{\alpha}} \right]$$

4. Find the Fourier transform of $f(t) = \frac{1}{\sqrt{2\pi}} \frac{2\alpha}{\alpha^2+t^2}$ with $\alpha > 0$

$$\mathrm{g}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2+t^2} e^{-i\omega t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\alpha}{(t-i\alpha)(t+i\alpha)} e^{-i\omega t} dt$$

The integrand has two poles: one at $t = i\alpha$ with residue $e^{\alpha\omega}/i$ and one at $t = -i\alpha$ with residue $e^{-\alpha\omega}/(-i)$. If $\omega > 0$, we must close the contour in the lower half plane, so contour enclosed only the pole at $t = -i\alpha$ in clockwise sense, so we get

$$\mathrm{g}(\omega) = \frac{1}{2\pi}(-2\pi i) \frac{e^{-\alpha\omega}}{-i} = e^{-\alpha\omega} \qquad (\omega > 0)$$

If $\omega < 0$, we must close the contour in the upper half plane, so contour enclosed only the pole at $t = i\alpha$, so we get

$$\mathrm{g}(\omega) = \frac{1}{\sqrt{2\pi}}(2\pi i) \frac{e^{\alpha\omega}}{i} = e^{\alpha\omega} \qquad (\omega < 0)$$

5. Find the Fourier transform of rectangular function $f(t) = \begin{cases} 1 & \text{if } |t| < a/2 \\ 0 & \text{if } |t| > a/2 \end{cases}$

$$\text{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{a/2} e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \frac{1}{-i\omega} \left(e^{-i\omega a/2} - e^{i\omega a/2}\right) = \sqrt{\frac{2}{\pi}} \frac{\sin(\omega a/2)}{\omega}$$

6. Find the Fourier transform of triangular Pulse $f(t) = \begin{cases} 1 - |t|/t_0 & \text{if } |t| < t_0 \\ 0 & \text{if } |t| > t_0 \end{cases}$

$$\text{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-t_0}^{0} (1 + \frac{t}{t_0}) e^{-i\omega t} dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{t_0} (1 - \frac{t}{t_0}) e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \Big[\frac{1}{-i\omega} (1 + \frac{t}{t_0}) e^{-i\omega t} \Big|_{-t_0}^{0} + \frac{1}{i\omega t_0} \int_{-t_0}^{0} e^{-i\omega t} dt + \frac{1}{-i\omega} (1 - \frac{t}{t_0}) e^{-i\omega t} \Big|_{0}^{t_0} - \frac{1}{i\omega t_0} \int_{0}^{t_0} e^{-i\omega t} dt \Big]$$

$$= \frac{1}{\sqrt{2\pi}} \Big[\frac{i}{\omega} + \frac{1}{\omega^2 x_0} (1 - e^{i\omega x_0}) - \frac{i}{\omega} - \frac{1}{\omega^2 x_0} (e^{-i\omega x_0} - 1) \Big] = \frac{2}{\sqrt{2\pi}} \Big[\frac{1 - \cos(\omega x_0)}{\omega^2 x_0} \Big]$$

$$= 2 \sqrt{\frac{2}{\pi}} \frac{1}{\omega^2 x_0} \sin^2(\omega x_0/2)$$

7. Find the Fourier transform of $f(t) = \cos(lt)$.

$$\text{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(lt) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ilt} + e^{-ilt}}{2} e^{-i\omega t} dt$$

$$= \frac{1}{2\sqrt{2\pi}} \big[\delta(k - l) + \delta(k + l) \big]$$

8. Find the Fourier transform of $f(t) = \sin(lt)$.

$$\text{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(lt) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ilt} - e^{-ilt}}{2i} e^{-i\omega t} dt$$

$$= \frac{1}{2i\sqrt{2\pi}} \big[\delta(k - l) - \delta(k + l) \big]$$

9. Find the Fourier transform of finite wave train $f(t) = \begin{cases} \sin \omega_0 t & \text{if } |t| < \frac{N\pi}{\omega_0} \\ 0 & \text{if } |t| > \frac{N\pi}{\omega_0} \end{cases}$

Since $f(t)$ is odd, we use the Fourier sine transformation

$$\text{g}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\frac{N\pi}{\omega_0}} \sin \omega_0 t \sin \omega t \, dt = \sqrt{\frac{2}{\pi}} \Big[\frac{\sin[(\omega_0 - \omega) N\pi/\omega_0]}{2(\omega_0 - \omega)} - \frac{\sin[(\omega_0 + \omega) N\pi/\omega_0]}{2(\omega_0 + \omega)} \Big]$$

It is clearly seen that large value of ω_0 and $\omega \simeq \omega_0$, only the first term will be of any importance because of the denominator. This is the amplitude for the single slit diffraction pattern. It has zeros at

$$\frac{\omega_0 - \omega}{\omega_0} = \frac{\Delta\omega}{\omega_0} = \pm\frac{1}{N}, \pm\frac{2}{N}, \cdots.$$

For large value N, $\text{g}_s(\omega)$ may be interpreted as proportional to Dirac delta distributive.

1.2 Dirac Delta Function

The Dirac δ-function has the property that [1]

$$\delta(t) = 0 \qquad \text{for } t \neq 0, \tag{11}$$

but its fundamental defining property is

$$\int f(t) \delta(t - a) dt = f(a) \tag{12}$$

provided the range of integration includes the point $t = a$; otherwise the integral equals zero. This leads immediately to two further useful results:

$$\begin{aligned} \int_{-a}^{b} \delta(t)dt &= 1 \quad \text{for all } a, b > 0 \qquad (13) \\ \int \delta(t-a) &= 1 \qquad (14) \end{aligned}$$

According to Fourier inversion theorem (Eq. 6)

$$\begin{aligned} f(t) = &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du \\ &= \int_{-\infty}^{\infty} du\, f(u) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-u)} d\omega \right) \qquad (15) \end{aligned}$$

Comparison of this with Eq. 12 shows that we may write the δ-function as

$$\delta(t-u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm i\omega(t-u)} d\omega \qquad (16)$$

Important Properties

$$\begin{aligned} \delta(t-a) &= \delta(a-t) \qquad (17) \\ f(t)\delta(t-a) &= f(a)\delta(t-a) \qquad (18) \\ \int_{-\infty}^{\infty} f(t)\delta'(t-a)dt &= -f'(a) \qquad (19) \\ \delta[c(t-a)] &= \frac{1}{|c|}\delta(t-a) \qquad (20) \\ \delta(t^2-a^2) &= \delta[(t-a)(t+a)] = \frac{1}{2a}[\delta(t-a)+\delta(t+a)], \quad a > 0 \qquad (21) \end{aligned}$$

1.3 Convolution Theorem

The convolution of the function $f(t)$ and $m(t)$ is defined by

$$f*m = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) m(t-u) du \qquad (22)$$

The convolution theorem state that the Fourier transformation of the convolution function of $f(t)$ and $m(t)$ is equal to the product of the Fourier transforms of $f(t)$ and $m(t)$, i.e.,

$$\mathcal{F}\{f*m\} = \mathcal{F}\{f\}\mathcal{F}\{m\} \qquad (23)$$

Proof:

$$\text{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du$$

and

$$n(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} m(v) e^{-i\omega v} dv$$

Then

$$\text{g}(\omega) n(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) m(v) e^{-i\omega(u+v)} du\, dv$$

Let $u + v = t$, then $dudv = \frac{\partial(u,v)}{\partial(u,t)} dudt = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial t} \end{vmatrix} dudt = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} dudt = dudt$

$$\therefore \quad \mathcal{F}\{f\}\mathcal{F}\{m\} = \text{g}(\omega) n(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) m(t-v) e^{-i\omega t} dudt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{f*m\} e^{-i\omega t} dt = \mathcal{F}\{f*m\}$$

Show that $f*m = m*f$
Let $t - u = v$

$$f*m = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)m(t-u)du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-v)m(v)dv = m*f$$

1.4 Properties of Fourier Transform

1.4.1 Differentiation

$$\mathcal{F}[f'(t)] = i\omega\,\mathrm{g}(\omega) \tag{24}$$

Proof:

$$\mathcal{F}[f'(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} f(t) e^{-i\omega t}\Big|_{-\infty}^{\infty} + i\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

But $\lim_{t\to\pm\infty} f(t) = 0$. This leads to

$$\mathcal{F}[f'(t)] = i\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = i\omega\,\mathrm{g}(\omega)$$

This process can be iterated for the nth derivative to yield

$$\mathcal{F}[f^{(n)}(t)] = (i\omega)^n\,\mathrm{g}(\omega) \tag{25}$$

1.4.2 Linearity

$$\mathcal{F}[af(t) + bm(t)] = a\mathcal{F}[f(t)] + b\mathcal{F}[m(t)] = a\,\mathrm{g}(\omega) + b\,n(\omega) \tag{26}$$

Proof:

$$\mathcal{F}[af(t) + bm(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \big(af(t) + bm(t)\big) e^{-i\omega t} dt$$

$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt + b\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} m(t) e^{-i\omega t} dt = a\mathcal{F}[f(t)] + b\mathcal{F}[m(t)] = a\,\mathrm{g}(\omega) + b\,n(\omega)$$

1.4.3 Translation

$$\mathcal{F}[f(t+a)] = e^{ia\omega}\mathrm{g}(\omega) \tag{27}$$

Proof:

$$\mathcal{F}[f(t+a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t+a) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-i\omega s} e^{i\omega a} ds = e^{ia\omega}\mathrm{g}(\omega)$$

1.4.4 Scaling

$$\mathcal{F}[f(at)] = \frac{1}{|a|}\mathrm{g}(\frac{\omega}{a}) \tag{28}$$

Proof:

$$\mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(at) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \frac{1}{|a|} \int_{-\infty}^{\infty} f(at) e^{-i\frac{\omega}{a}at} d(at) = \frac{1}{|a|}\mathrm{g}(\frac{\omega}{a})$$

1.4.5 Exponential multiplication

$$\mathcal{F}[e^{\alpha t} f(t)] = \mathrm{g}(\omega + i\alpha) \tag{29}$$

Proof:

$$\mathcal{F}[e^{\alpha t} f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\alpha t} f(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i(\omega + i\alpha)t} dt = \mathrm{g}(\omega + i\alpha)$$

1.4.6 Complex Conjugation

$$\mathcal{F}[f^*(t)] = \mathrm{g}^*(-\omega) \tag{30}$$

Proof:
Taking inverse Fourier transform, we get

$$f^*(t) = \Big[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{g}(\omega)e^{i\omega t}d\omega\Big]^* = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{g}^*(\omega)e^{-i\omega t}d\omega$$

Replacing $\omega = -\omega'$, we get

$$f^*(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{g}^*(-\omega')e^{i\omega' t}d\omega' = \mathcal{F}^{-1}[g^*(-\omega)]$$

$$\therefore \quad \mathcal{F}[f^*(t)] = \mathrm{g}^*(-\omega)$$

2 Three dimensional Fourier transforms

Three dimensional Fourier transforms

$$\mathrm{g}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{-i\,\mathbf{k}\cdot\mathbf{r}}d^3r \tag{31}$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{g}(\mathbf{k})e^{i\,\mathbf{k}\cdot\mathbf{r}}d^3k \tag{32}$$

The three-dimensional Dirac δ-function can be written as

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3}\int e^{i\,\mathbf{k}\cdot\mathbf{r}}d^3k \tag{33}$$

2.1 Examples

1. In three-dimensional space a function f($\mathbf{r}$) possesses spherical symmetry, so that f($\mathbf{r}$) = f(r). Find the Fourier transform of f($\mathbf{r}$) as a one-dimensional integral.
Since f($\mathbf{r}$) is spherically symmetric, we then have $d^3r = r^2\sin\theta dr d\theta d\phi$ and $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$.The Fourier transform is given by

$$\mathrm{g}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{-i\,\mathbf{k}\cdot\mathbf{r}}d^3r = \frac{1}{(2\pi)^{3/2}}\int_0^{\infty}dr\int_0^{\pi}d\theta\int_0^{2\pi}d\phi f(r)r^2\sin\theta e^{-i\,kr\cos\theta}$$

$$= \frac{1}{(2\pi)^{3/2}}\int_0^{\infty}dr\, 2\pi f(r)r^2\Big(\frac{e^{-i\,kr\cos\theta}}{i\,kr}\Big)\Big|_{\theta=0}^{\pi} = \frac{1}{(2\pi)^{3/2}}\int_0^{\infty}4\pi r^2 f(r)\Big(\frac{\sin kr}{kr}\Big)dr$$

2. Fourier transform of the Yukawa potential $f(r) = \frac{e^{-\alpha r}}{r}$

$$\mathrm{g}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{-i\,\mathbf{k}\cdot\mathbf{r}}d^3r = \frac{1}{(2\pi)^{3/2}}\int_0^{\infty}4\pi r^2\frac{e^{-\alpha r}}{r}\Big(\frac{\sin kr}{kr}\Big)dr$$

$$= \frac{4\pi}{(2\pi)^{3/2}}\int_0^{\infty}\frac{e^{i(k+i\alpha)r} - e^{-i(k-i\alpha)r}}{2ik}dr = \frac{4\pi}{(2\pi)^{3/2}}\Big[-\frac{1}{i(k+i\alpha)} - \frac{1}{i(k-i\alpha)}\Big]\frac{1}{2ik}$$

$$= \frac{1}{(2\pi)^{3/2}}\frac{4\pi}{k^2+\alpha^2}$$

3. Fourier transform of Coulomb potential $f(r) = \frac{1}{r}$

$$\mathrm{g}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{-i\,\mathbf{k}\cdot\mathbf{r}}d^3r = \frac{1}{(2\pi)^{3/2}}\int_0^{\infty}4\pi r^2\frac{1}{r}\Big(\frac{\sin kr}{kr}\Big)dr$$

$$= \frac{4\pi}{(2\pi)^{3/2}}\int_0^{\infty}\frac{e^{ikr} - e^{-ikr}}{2ik}dr = \frac{4\pi}{(2\pi)^{3/2}}\Big[-\frac{1}{ik} - \frac{1}{ik}\Big]\frac{1}{2ik} = \frac{1}{(2\pi)^{3/2}}\frac{4\pi}{k^2}$$

4. Fourier transform of 3-D Gaussian wave $f(r) = e^{-\alpha r^2}$

$$\mathrm{g}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{-i\,\mathbf{k}\cdot\mathbf{r}}d^3r = \frac{1}{(2\pi)^{3/2}}\int_0^{\infty}4\pi r^2 e^{-\alpha r^2}\Big(\frac{\sin kr}{kr}\Big)dr = \frac{1}{2\alpha}e^{-k^2/4\alpha}$$

3 Wave Equation

The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{34}$$

where v is the phase velocity of the wave propagation and $y(x,t)$ is the solution. The initial conditions are

$$y(x,0) = f(x), \qquad \frac{\partial y(x,t)}{\partial t}\Big|_{t=0} = 0 \tag{35}$$

where f is assumed localized, meaning that $\lim_{x=\pm\infty} f(x) = 0$.

We take the Fourier transforms in x of Eq. (34) by using α as the transform variable

$$\int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial x^2} e^{-i\alpha x} dx = \frac{1}{v^2}\int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2} e^{-i\alpha x} dx \tag{36}$$

If we consider

$$Y(\alpha,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(x,t)e^{-i\alpha x}dx \quad \text{or} \quad y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Y(\alpha,t)e^{i\alpha x}d\alpha \tag{37}$$

as the transform of the solution $y(x,t)$ of our PDE, we can rewrite Eq. (36) as

$$(i\alpha)^2 Y(\alpha,t) = \frac{1}{v^2}\frac{\partial^2 Y(\alpha,t)}{\partial t^2} \tag{38}$$

Taking the Fourier transforms of the Eq. (35), we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(x,0)e^{-i\alpha x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx = Y(\alpha,0) = F(\alpha) \tag{39}$$

$$\frac{\partial Y(\alpha,t)}{\partial t}\Big|_{t=0} = 0 \tag{40}$$

Solving Eq. (38) subject to the initial condition on Y given in Eq. (40), we obtain

$$Y(\alpha,t) = F(\alpha)\frac{e^{i\alpha vt} + e^{-i\alpha vt}}{2} \tag{41}$$

If we apply the inverse Fourier transforms in Eq. (41)

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Y(\alpha,t)e^{i\alpha x}d\alpha = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)\frac{e^{i\alpha(x+vt)} + e^{i\alpha(x-vt)}}{2} \tag{42}$$

The left-hand side of Eq. (42) is clearly $y(x,t)$; each term on the right-hand side is the inverse Fourier transform of F and is it f. This leads to

$$y(x,t) = \frac{1}{2}\Big[f(x+vt) + f(x-vt)\Big]. \tag{43}$$

This is the required solution of the wave equation.

4 Heat Flow PDE

The one-dimension heat flow PDE is

$$\frac{\partial \psi}{\partial t} = a^2\frac{\partial^2 \psi}{\partial x^2} \tag{44}$$

where $\psi(x,t)$ is the temperature at position x and t.

We transform the x dependence, with the transform variable denoted α, writing the transform of $\psi(x,t)$ as $\Psi(\alpha,t)$, and we get

$$\frac{\partial \Psi(\alpha,t)}{\partial t} = -a^2\alpha^2\Psi(\alpha,t) \tag{45}$$

with the general solution is

$$\ln \Psi(\alpha, t) = -a^2\alpha^2 t + \ln C(\alpha), \quad \text{or} \quad \Psi(\alpha, t) = C(\alpha)e^{-a^2\alpha^2 t} \tag{46}$$

The physical significance of $C(\alpha)$ is that it is the initial spatial distribution of Ψ or, the Fourier transform of the initial temperature profile $\psi(x, 0)$. If we consider $C(\alpha)$ is constant i.e. C and we take the inverse Fourier transform

$$\begin{aligned} \psi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C\, e^{-a^2\alpha^2 t} e^{i\alpha x} d\alpha \\ &= \frac{C}{a\sqrt{2t}} e^{-\frac{x^2}{4a^2 t}} \end{aligned} \tag{47}$$

5 More Examples

1. Find the Fourier transform of $f(t) = \begin{cases} 1 & \text{if } |t| < 1 \\ 0 & \text{if } |t| > 1 \end{cases}$ Hence calculate $\int_0^\infty \frac{\sin\omega}{\omega}\, d\omega$.

$$\mathrm{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}} \frac{1}{-i\omega}\left(e^{-i\omega} - e^{i\omega}\right) = \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}$$

Using the inverse formula, we get

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega)e^{i\omega t}d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega} e^{i\omega t}d\omega = \begin{cases} 1 & \text{if } |t| < 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

Putting $t = 0$, we have

$$\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \pi \quad \Rightarrow \quad \int_0^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \frac{\pi}{2}$$

2. Find the Fourier transform of $f(t) = \begin{cases} 1 - t^2 & \text{if } |t| \le 1 \\ 0 & \text{if } |t| > 1 \end{cases}$

Hence calculate $\int_0^\infty \frac{\omega\cos\omega - \sin\omega}{\omega^3} \cos\frac{\omega}{2}\, d\omega$.

$$\mathrm{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - t^2)e^{-i\omega t}dt$$

$$= \frac{1}{\sqrt{2\pi}} e^{-i\omega t}\Big[\frac{1-t^2}{-i\omega} - \frac{2t}{\omega^2} + \frac{2}{-i\omega^3}\Big]\Bigg|_{-1}^{1} = \frac{1}{\sqrt{2\pi}}\Big[-\frac{4}{\omega^2}\cos\omega + \frac{4}{\omega^3}\sin\omega\Big]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{4}{\omega^3}\big[\sin\omega - \omega\cos\omega\big]$$

Using the inverse formula, we get

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega)e^{i\omega t}d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{4}{\omega^3}\big[\sin\omega - \omega\cos\omega\big]e^{i\omega t}d\omega = \begin{cases} 1 - t^2 & \text{if } |t| \le 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

Putting $t = \frac{1}{2}$, we have

$$\int_{-\infty}^{\infty} \frac{\omega\cos\omega - \sin\omega}{\omega^3} e^{i\omega/2}d\omega = -\frac{3}{4} \times \frac{\pi}{2}$$

If we take the real part of the above equation, we have

$$\int_0^{\infty} \frac{\omega\cos\omega - \sin\omega}{\omega^3} \cos(\omega/2)d\omega = -\frac{3\pi}{16} \qquad \text{[for even integral]}$$

3. Find the Fourier sine transform of $f(t) = e^{-|t|}$. Hence show that $\int_0^\infty \frac{\omega \sin m\omega}{1+\omega^2} d\omega = \frac{\pi}{2} e^{-m},\ m > 0$.

In the interval $(0, \infty)$, t is positive so, $e^{-|t|} = e^{-t}$. Using the Fourier sine transformation,

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin(\omega t)\, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t} \sin(\omega t)\, dt$$

$$= \sqrt{\frac{2}{\pi}} \times -\frac{e^{-t}}{1+\omega^2} \left[\sin(t\omega) + \omega \cos(t\omega) \right] \Big|_0^\infty = \sqrt{\frac{2}{\pi}} \frac{\omega}{1+\omega^2}$$

Using the inverse formula for Fourier sine transformation, we get

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\omega) \sin(\omega t)\, d\omega = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\omega}{1+\omega^2} \sin(\omega t)\, d\omega$$

Changing t to x, we get

$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{\omega \sin m\omega}{1+\omega^2}\, d\omega \quad \Rightarrow \quad \int_0^\infty \frac{\omega \sin m\omega}{1+\omega^2} d\omega = \frac{\pi}{2} e^{-m}$$

4. Find the Fourier sine transform of $f(t) = \frac{e^{-\alpha t}}{t}$
   Using the Fourier sine transformation

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin(\omega t)\, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-\alpha t}}{t} \sin(\omega t)\, dt$$

Differentiating both side with respect to ω, we get

$$g_s'(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{t\, e^{-\alpha t}}{t} \cos(\omega t)\, dt = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2+\omega^2}$$

Now

$$g_s(\omega) = \int \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2+\omega^2} d\omega = \sqrt{\frac{2}{\pi}} \tan^{-1} \left(\frac{\omega}{\alpha}\right)$$

5. Find the Fourier sine and cosine transform of $te^{-\alpha t}$.
   Using the Fourier sine transformation

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin(\omega t)\, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\alpha t} \sin(\omega t)\, dt$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2+\omega^2} \left[e^{-\alpha t} (-\alpha \sin(\omega t) - \omega \cos(\omega t)) \right] \Bigg|_0^\infty = \sqrt{\frac{2}{\pi}} \frac{\omega}{\alpha^2+\omega^2}$$

and the Fourier sine transformation

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos(\omega t)\, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\alpha t} \cos(\omega t)\, dt$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2+\omega^2} \left[e^{-\alpha t} (-\alpha \cos(\omega t) + \omega \sin(\omega t)) \right] \Bigg|_0^\infty = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2+\omega^2}$$

Now

$$\mathcal{F}_s[te^{-\alpha t}] = -\frac{d}{d\omega} g_c(\omega) = -\frac{d}{d\omega} \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2+\omega^2} = \sqrt{\frac{2}{\pi}} \frac{2\alpha\omega}{(\alpha^2+\omega^2)^2}$$

and

$$\mathcal{F}_c[te^{-\alpha t}] = \frac{d}{d\omega} g_s(\omega) = \frac{d}{d\omega} \sqrt{\frac{2}{\pi}} \frac{\omega}{\alpha^2+\omega^2} = \sqrt{\frac{2}{\pi}} \frac{\alpha^2-\omega^2}{(\alpha^2+\omega^2)^2}$$

References

[1] K. F. Riley, M. P. Hobson and S. J. Bence, Cambridge University Press, ISBN-13 978-0-521-67971-8.

[2] Arfken, Weber and Harris, Elsevier, ISBN: 978-93-81269-55-8.