Assignment: 1
Mathematical Physics-III: Complex Variables

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1. Answer the following questions: 2 Marks

(i) Show that $|z_1 + z_2| \leq |z_1| + |z_2|$.

(ii) Show that $|z_1 \pm z_2| \geq |z_1| - |z_2|$.

(iii) Show that $2 + i = \sqrt{5}\, e^{i \tan^{-1}(1/2)}$.

(iv) If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, prove that

$$z_1 z_2 = r_1 r_2\big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big).$$

(v) What do you mean by poles and non-isolated singularities of a function?

(vi) Find the angle θ for which $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is imaginary.

(vii) Find the value of $(-1)^i$.

(viii) If n is a positive integer, proved that $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6.}$.

(ix) Show that $f(z) = z^*$ is not differentiable at any point in the complex plane.

(x) show that $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \frac{-1+\sqrt{3}i}{2}$.

(xi) Prove that $1 + \cos 72^0 + \cos 144^0 + \cos 216^0 + \cos 288^0 = 0$.

(xii) Prove that $\cos 36^0 + \cos 72^0 + \cos 108^0 + \cos 144^0 = 0$.

(xiii) Find an equation for a circle of radius 2 with center (-3,4).

(xiv) Evaluate $\lim_{z\to i} \frac{z^2+1}{z^6+1}$.

(xv) what do you mean by removable and essential singularities of the function $f(z)$?

(xvi) Discuss the nature of singularity for the function $z\exp(\frac{1}{z})$.

(xvii) State the Cauchy-Riemann conditions.

(xviii) Define simply and multiply connected region.

(xix) State and explain Cauchy's integral formula.

(xx) State and explain the Taylor's theorem.

(xxi) State and explain the Laurent's theorem.

(xxii) Expand e^{-z} in a Taylor series about $z = 0$.

(xxiii) State the Residue theorem.

(xxiv) Find the residues of $f(z) = \frac{2z+1}{z^2-z-2}$.

(xxv) Find the residues of $f(z) = e^{-1/z}\sin(1/z)$.

2. Answer the following questions: 4 Marks

(i) State and prove the De Moivre's Theorem.

(ii) Find the value of (a) $\cos 5\theta$ (b) $\sin^3\theta$.

(iii) Find the 5th roots of $1 + i$.

(iv) Solve the equations (a) $z^4 + 256 = 0$ and (b) $z^4 - 4z^2 + 4 - 2i = 0$.

(v) Show that $\frac{\sin\theta}{2} + \frac{\sin 2\theta}{2^2} + \frac{\sin 3\theta}{2^3} + \cdots = \frac{2\sin\theta}{5-4\cos\theta}$.

(vi) Proved that $\sin^{-1} z = \frac{1}{i}\ln(iz + \sqrt{1-z^2})$.

(vii) Evaluate $\lim_{z\to 0}\left(\frac{\sin z}{z}\right)^{1/z^2}$.

(viii) Locate and classify the singularities of the functions (a) $\frac{\ln(z-2)}{(z^2+2z+2)^4}$ and (b) $\frac{\sin\sqrt{z}}{\sqrt{z}}$.

(ix) Verify that the Cauchy-Riemann equations are satisfied for the functions: (a) $\ln z$ and (b) ze^{-z}.

(x) Derive the Cauchy-Riemann conditions in polar coordinate.

(xi) Prove that $u = e^x(x\cos y - y\sin y)$ is harmonic. Find the corresponding value of $f(z) = u + iv$.

(xii) If $f(z) = u + iv$ is analytic in the region $\mathcal{R}$, prove that u and v are harmonic in $\mathcal{R}$ if they have continuous second partial derivatives in $\mathcal{R}$.

(xiii) If $f(z)$ be analytic in a region bounded by two simple closed curves C and C_1 (where C_1 lies inside C) and on these curve, then prove that $\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$.

(xiv) Verify Cauchy's theorem for the function $z^3 - iz^2 - 5z + 2i$ if C is (a) the circle $|z| = 1$, (b) the circle $|z-1| = 2$.

(xv) State and prove the Cauchy's integral formula.

(xvi) If $f(z)$ be analytic inside and on the boundary of a simply connected region $\mathcal{R}$, prove that $f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz$.

(xvii) Using Cauchy's integral formula evaluate $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z^2+1)^2} dz$ if $t > 0$ and C is the circle $|z| = 3$.

(xviii) Using Cauchy's integral formula evaluate $\oint_C \frac{z-3}{z^2+2z+3} dz$, where C is the circle $|z+1-i| = 2$.

(xix) Expand $f(z) = \cos z$ in a Taylor series about $z = \pi/4$.

(xx) Find the pole and residue at each pole of the function $f(z) = \frac{z}{(z^2+2)(z-1)^2}$.

(xxi) Evaluate $\oint_C \frac{2+3\sin \pi z}{z(z-1)^2} dz$, where C is square having vertices at $3+3i$, $3-3i$, $-3+3i$, $-3-3i$.

(xxii) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$.

(xxiii) Show that $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ given $a > |b|$.

(xxiv) Evaluate $\int_0^{2\pi} \frac{\sin 3\theta}{5-4\cos\theta} d\theta$.

(xxv) Show that $\int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$.